[9]Comlekci, T., "A General Inverse Transformation Method for Isoparametric Elements," private communication, 1995.
[10]Pagano, N., "Exact Solutions for Composite Laminates in Cylindrical Bending," *Journal of Composite Materials*, Vol. 3, 1969, pp. 398–411.
[11]Pagano, N., and Hatfield, S., "Elastic Behavior of Multilayered Bidirectional Composites," *AIAA Journal*, Vol. 10, No. 7, 1972, pp. 931–933.
[12]Guadenzi, P., Barboni, P., and Mannini, A., "A Finite Element Evaluation of Single-Layer and Multi-Layer Theories for the Analysis of Laminated Plates," *Composite Structures*, Vol. 30, No. 4, 1995, pp. 427–440.

Iterative Calculation of the Transverse Shear Distribution in Laminated Composite Beams

J. A. Zapfe* and G. A. Lesieutre†
Pennsylvania State University,
University Park, Pennsylvania 16802

Introduction

CLASSICAL lamination theory (CLT) can often be used to accurately predict the stress distribution in beams fabricated from composite materials. However, when the beam is short or when it contains shear soft materials, nonclassical effects such as transverse shear deformation must be considered. One group of beam models that account for transverse shear deformation is known as smeared laminate models (SLMs). SLMs assume a form of the displacement field through the entire thickness of the beam and result in a set of equivalent properties (like the [A], [B], and [D] matrices in CLT) that are used to determine the gross behavior of the beam (e.g., deflection and natural frequency). SLMs that have been presented in the literature lead to constant shear strain, quadratic shear strain, constant shear stress, and quadratic shear stress.[1–6] The limitation of smeared laminate models, when applied to general laminate configurations, is the inherent assumption about the stress distribution that follows from the assumption of the displacement field. If the assumed stress distribution is not representative of the actual stress field, then the smeared laminate model can be significantly in error.

The first SLM that could accurately predict the stress distribution in general laminates was presented by Vijayakumar and Krishna Murty.[7] The authors used an iterative process to successively refine the estimate of the stress/strain field in the laminate. Vijayakumar and Krishna Murty's method produced excellent results when compared with exact solutions. The present SLM is also an iterative method, but unlike Vijayakumar and Kirshna Murty's method, the present model uses the assumed displacement approach of a traditional SLM.

Basic Theory

Consider a beam in which x and u are the in-plane coordinate and displacement, z and w are the transverse coordinate and displacement, and h_u and h_l locate the upper and lower surfaces of the beam. The assumed displacement field over the domain of the beam is

$$u(x,z) = u_0(x) - z\frac{\partial w(x)}{\partial x} + f(z)g(x)$$
$$w(x,z) = w(x) \tag{1}$$

The first two terms in the in-plane displacement expression define the CLT displacement field for a beam. The last term, $f(z)g(x)$, can be thought of as a correction to account for transverse shear effects. The function $f(z)$ represents the shape of the correction through the thickness, whereas $g(x)$ determines its distribution along the beam. The goal of the iterative process is the determination of the function $f(z)$ that makes the stresses and strains self-consistent.

The strain field provides more insight into the nature of the function $f(z)$. Applying the linear strain-displacement relations to Eq. (1) yields

$$\epsilon_x(x,z) = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + f(z)\frac{\partial g}{\partial x}$$
$$\gamma_{xz}(x,z) = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\partial f}{\partial z}g(x) \tag{2}$$

From Eq. (2), it can be seen that the gradient $\partial f/\partial z$ represents the shape of the shear strain field through the thickness at a given x location. Therefore, if the shape of the shear strain is known, $f(z)$ can be estimated by integrating the strain through the thickness. In general, $f(z)$ has no closed-form solution. In the present formulation, $f(z)$ is represented numerically as a tabular function.

Static Simply Supported Beams

The present analysis pertains to simply supported beams under a transverse load of the form

$$q(x) = q_0\sin(kx), \qquad k = n\pi/L \tag{3}$$

where k is the wave number and n is the number of $\frac{1}{2}$ sine waves along the length of the beam.

Whereas the shear correction $f(z)$ changes from iteration to iteration, at any given iteration, $f(z)$ is prescribed and can be treated as a known function. Differential equations of equilibrium and boundary conditions can be developed like any other SLM. The equilibrium equations are

$$-K_1u_0'' + K_2w''' - K_3g'' = 0$$
$$-K_2u_0''' + K_4w^{iv} - K_5g''' = q(x) \tag{4}$$
$$-K_3u_0'' + K_5w''' - K_6g'' + K_7g = 0$$

and the simply supported boundary conditions, specified at $x = 0$ and $x = L$, are given by

$$K_1u_0' - K_2w'' + K_3g' = 0$$
$$K_3u_0' - K_5w'' + K_6g' = 0$$
$$w = 0 \tag{5}$$
$$-K_2u_0' + K_4w'' - K_5g' = 0$$

The terms K_{1-7} are section stiffness parameters given by

$$K_{[1,2,3,4,5,6]} = b\int_{h_l}^{h_u} E(z)[1, z, f(z), z^2, zf(z), f^2(z)]\,dz$$
$$K_7 = b\int_{h_l}^{h_u} G(z)\left[\frac{\partial f(z)}{\partial z}\right]^2 dz \tag{6}$$

In the present formulation, the integrals are evaluated numerically using a trapezoidal method.

Functions that satisfy the differential equations and boundary conditions are

$$u_0(x) = U_0\cos(kx), \qquad w(x) = W_0\sin(kx)$$
$$g(x) = G_0\cos(kx) \tag{7}$$

Received Feb. 28, 1995; presented as Paper 95-1501 at the AIAA/ASME/ASCE/AHS/ASC 36th Structures, Structural Dynamics, and Materials Conference, New Orleans, LA, April 10–12, 1995; revision received Nov. 25, 1995; accepted for publication Dec. 30, 1995.

*Ph.D. Candidate, Department of Aerospace Engineering; currently Research Engineer, Kinetic Systems, Inc., Roslindale, MA 02131. Member AIAA.
†Associate Professor, Department of Aerospace Engineering. Associate Fellow AIAA.

Substitution of Eqs. (7) into Eqs. (4) leads to a matrix equation for the coefficients (U_0, W_0, G_0):

$$\begin{bmatrix} K_1k^2 & -K_2k^3 & K_3k^2 \\ K_2k^3 & K_4k^4 & -K_5k^3 \\ K_3k^2 & -K_5k^3 & (K_6k^2 + K_7) \end{bmatrix} \begin{bmatrix} U_0 \\ W_0 \\ G_0 \end{bmatrix} = \begin{bmatrix} 0 \\ q_0 \\ 0 \end{bmatrix} \tag{8}$$

Improved Estimate for Shear Correction

The new estimate for the shear correction function $f(z)$ is obtained from the shear stress distribution as calculated from the equation of elemental stress equilibrium,

$$\frac{\partial \tau_{xz}}{\partial z} = -\frac{\partial \sigma_x}{\partial x} \tag{9}$$

The normal stress gradient is known from the solution to Eq. (8), and so the shear gradient is

$$\frac{\partial \tau_{xz}}{\partial z} = -E(z)\left[-k^2U_0 + k^3zW_0 - k^2f(z)G_0\right]\cos(kx) \tag{10}$$

The shear gradient has a separable form that means the shape of the gradient through the thickness is the same at each x location. The shape of the shear stress is then given by

$$\tau_{xz}(z) = \int_{h_l}^{z} -E(z)\left[-k^2U_0 + k^3zW_0 - k^2f(z)G_0\right]\mathrm{d}z \tag{11}$$

The shape of the shear strain distribution is derived using the constitutive relation

$$\gamma_{xz}(z) = \frac{\tau_{xz}(z)}{G(z)} \tag{12}$$

The new estimate of $f(z)$ is obtained by integrating the shear strain through the thickness

$$f(z) = \int_{h_l}^{z} \gamma_{xz}(z)\,\mathrm{d}z + F_0 \tag{13}$$

where the constant F_0 ensures that $f(z=0)=0$. As with the other section integrals, Eq. (13) is evaluated numerically using a trapezoidal method. The new estimate of $f(z)$ is used as the shear correction in the next iteration. The solution steps for subsequent iterations are identical.

The present analysis uses the first-order shear deformation theory displacement field $[f(z) = z]$ for the first iteration; however, almost any reasonable displacement assumption can be used.

Results: Comparison to Exact Solutions

Pagano[8] presented a set of exact elasticity solutions for simply supported cross-ply laminates in cylindrical bending under sinusoidal transverse load. The iterative SLM (ISLM) was compared with Pagano's solution for case 3 (0, 90, 0). Zapfe[9] presents results for additional cases as well as a detailed examination of convergence and computational speed.

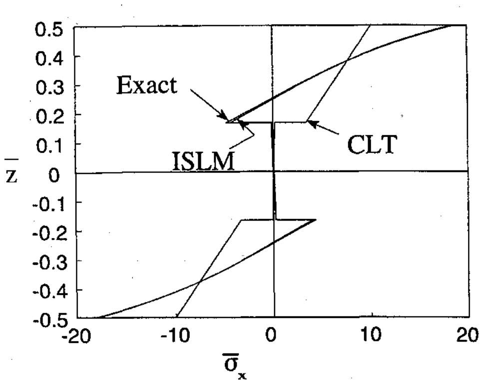

Fig. 1 Pagano case 3: in-plane normal stress.

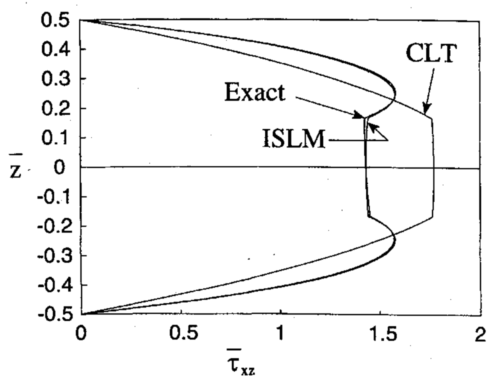

Fig. 2 Pagano case 3: transverse shear stress.

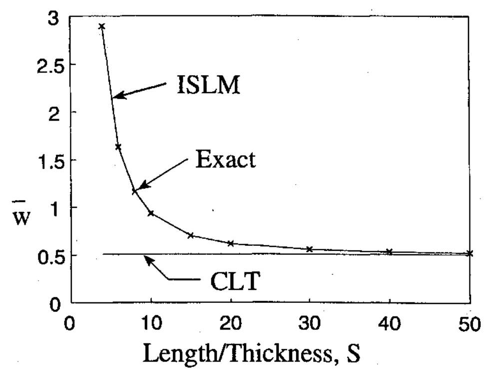

Fig. 3 Pagano case 3: transverse deflection.

Figure 1 shows the in-plane normal stress distribution obtained using the exact solution, the ISLM, and CLT. The ISLM results correspond to the solution after 10 iterations. The ISLM stress distribution matches the exact solution, which is markedly different from CLT. The transverse shear stress distribution appears in Fig. 2. The slight variation between the ISLM and exact solution is caused by transverse normal strain, which is not included in the present iterative model. Figure 3 shows the midspan deflection variation with beam length. The iterative results match the exact solution, and both solutions tend toward CLT as the beam becomes long and thin.

Conclusion

A smeared laminate beam model has been presented that can accurately predict the stress/strain distribution in general laminated beams, a capability that has not been previously possible with conventional smeared laminate beam models.

References

[1] Yang, P. C., Norris, C. H., and Stavsky, Y., "Elastic Wave Propagation in Heterogeneous Plates," *International Journal of Solids and Structures*, Vol. 2, 1966, pp. 665–684.

[2] Krishna Murty, A. V., "Vibrations of Short Beams," *AIAA Journal*, Vol. 8, No. 1, 1970, pp. 34–38.

[3] Levinson, M., "A New Rectangular Beam Theory," *Journal of Sound and Vibration*, Vol. 74, Jan. 1981, pp. 81–87.

[4] Reddy, J. N., "A Simple Higher-Order Theory for Laminated Composite Plates," *Journal of Applied Mechanics*, Vol. 51, Dec. 1984, pp. 745–752.

[5] Di Sciuva, M., "Bending, Vibration and Buckling of Simply Supported Thick Multilayered Orthotropic Plates: An Evaluation of a New Displacement Model," *Journal of Sound and Vibration*, Vol. 105, No. 3, 1986, pp. 425–442.

[6] Bhaskar, K., and Varadan, T. K., "Refinement of Higher-Order Laminated Plate Theories," *AIAA Journal*, Vol. 27, No. 12, 1989, pp. 1830, 1831.

[7] Vijayakumar, K., and Krishna Murty, A. V., "Iterative Modeling for Stress Analysis of Composite Laminates," *Composites Science and Technology*, Vol. 32, 1988, pp. 165–181.

[8] Pagano, N. J., "Exact Solutions for Composite Laminates in Cylindrical Bending," *Journal of Composite Materials*, Vol. 3, July 1969, pp. 398–411.

[9] Zapfe, J. A., "Prediction of Damping in Laminated Beams Using a Discrete Layer Finite Element Model and an Iterative Smeared Laminate Model," Ph.D. Thesis, Dept. of Aerospace Engineering, Pennsylvania State Univ., University Park, PA, 1995.